A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail*

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February 2, 2008

AMS subject classification: Primary 60G70 Secondary 60G15

Short Title: Distribution of the Maximum.

Key words and phrases: Gaussian fields, Rice Formula, Euler-Poincaré Characteristic, Distribution of the Maximum, Density of the Maximum, Random Matrices.

Abstract

We study the probability distribution $F(u)$ of the maximum of smooth Gaussian fields defined on compact subsets of $\mathbb{R}^d$ having some geometric regularity.

Our main result is a general expression for the density of F. Even though this is an implicit formula, one can deduce from it explicit bounds for the density, hence for the distribution, as well as improved expansions for $1 - F(u)$ for large values of u.

The main tool is the Rice formula for the moments of the number of roots of a random system of equations over the reals.

This method enables also to study second order properties of the expected Euler Characteristic approximation using only elementary arguments and to extend these kind of results to some interesting classes of Gaussian fields. We obtain more precise results for the "direct method" to compute the distribution of the maximum, using spectral theory of GOE random matrices.

1 Introduction and notations

Let $\mathcal{X} = \{X(t) : t \in S\}$ be a real-valued random field defined on some parameter set S and $M := \sup_{t\in S} X(t)$ its supremum.

The study of the probability distribution of the random variable M, i.e. the function $F_M(u) := \mathrm{P}\{M \le u\}$ is a classical problem in probability theory. When the process is Gaussian, general inequalities allow to give bounds on $1 - F_M(u) = \mathrm{P}\{M > u\}$ as well as asymptotic results for $u \to +\infty$. A partial account of this well established theory, since the founding paper by Landau and Shepp [20] should contain - among a long list of contributors - the works of Marcus and Shepp [24], Sudakov and Tsirelson [30], Borell [13] [14], Fernique [17], Ledoux and Talagrand [22], Berman [11] [12], Adler[2], Talagrand [32] and Ledoux[21].

During the last fifteen years, several methods have been introduced with the aim of obtaining more precise results than those arising from the classical theory, at least under certain restrictions on the process $\mathcal{X}$, which are interesting from the point of view of the mathematical theory as well as in many significant applications. These restrictions include the requirement

*This work was supported by ECOS program U03E01.

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the domain S to have certain finite-dimensional geometrical structure and the paths of the random field to have a certain regularity.

Some examples of these contributions are the double sum method by Piterbarg [28]; the Euler-Poincaré Characteristic (EPC) approximation, Taylor, Takemura and Adler [34], Adler and Taylor [3]; the tube method, Sun [31] and the well- known Rice method, revisited by Azaïs and Delmas [5], Azaïs and Wschebor [6]. See also Rychlik [29] for numerical computations.

The results in the present paper are based upon Theorem 3 which is an extension of Theorem 3.1 in Azaïs and Wschebor [8] allowing to express the density p_M of F_M by means of a general formula. Even though this is an exact formula, it is only implicit as an expression for the density, since the relevant random variable M appears in the right-hand side. However, it can be usefully employed for various purposes.

First, one can use Theorem 3 to obtain bounds for $p_M(u)$ and thus for $\mathrm{P}\{M > u\}$ for **every** u by means of replacing some indicator function in (4) by the condition that the normal derivative is "extended outward" (see below for the precise meaning). This will be called the "direct method". Of course, this may be interesting whenever the expression one obtains can be handled, which is the actual situation when the random field has a law which is stationary and isotropic. Our method relies on the application of some known results on the spectrum of random matrices.

Second, one can use Theorem 3 to study the asymptotics of $\mathrm{P}\{M > u\}$ as $u \to +\infty$. More precisely, one wants to write, whenever it is possible

$$\mathrm{P}\{M > u\} = A(u) \ \exp\big(-\frac{1}{2}\frac{u^2}{\sigma^2}\big) \ + \ B(u) \tag{1}$$

where $A(u)$ is a known function having polynomially bounded growth as $u \to +\infty$, $\sigma^2 = \sup_{t\in S} \mathrm{Var}(X(t))$ and $B(u)$ is an error bounded by a centered Gaussian density with variance σ_1^2, $\sigma_1^2 < \sigma^2$. We will call the first (respectively the second) term in the right-hand side of (1) the "first (resp second) order approximation of $\mathrm{P}\{M > u\}$."

First order approximation has been considered in [3] [34] by means of the expectation of the EPC of the excursion set $E_u := \{t \in S : X(t) > u\}$. This works for large values of u. The same authors have considered the second order approximation, that is, how fast does the difference between $\mathrm{P}\{M > u\}$ and the expected EPC tend to zero when $u \to +\infty$.

We will address the same question both for the direct method and the EPC approximation method. Our results on the second order approximation only speak about the size of the variance of the Gaussian bound. More precise results are only known to the authors in the special case where S is a compact interval of the real line, the Gaussian process $\mathcal{X}$ is stationary and satisfies a certain number of additional requirements (see Piterbarg [28] and Azaïs et al. [4]).

Theorem 5 is our first result in this direction. It gives a rough bound for the error $B(u)$ as $u \to +\infty$, in the case the maximum variance is attained at some strict subset of the face in S having the largest dimension. We are not aware of the existence of other known results under similar conditions.

In Theorem 6 we consider processes with constant variance. This is close to Theorem 4.3 in [34]. Notice that Theorem 6 has some interest only in case $\sup_{t\in S} \kappa_t < \infty$, that is, when one can assure that $\sigma_1^2 < \sigma^2$ in (1). This is the reason for the introduction of the additional hypothesis $\kappa(S) < \infty$ on the geometry of S, (see below (64) for the definition of $\kappa(S)$), which is verified in some relevant situations (see the discussion before the statement of Theorem 6).

In Theorem 7, S is convex and the process stationary and isotropic. We compute the exact asymptotic rate for the second order approximation as $u \to +\infty$ corresponding to the direct

method.

In all cases, the second order approximation for the direct method provides an upper bound for the one arising from the EPC method.

Our proofs use almost no differential geometry, except for some elementary notions in Euclidean space. Let us remark also that we have separated the conditions on the law of the process from the conditions on the geometry of the parameter set.

Third, Theorem 3 and related results in this paper, in fact refer to the density p_M of the maximum. On integration, they imply immediately a certain number of properties of the probability distribution F_M, such as the behaviour of the tail as $u \to +\infty$.

Theorem 3 implies that F_M has a density and we have an implicit expression for it. The proof of this fact here appears to be simpler than previous ones (see Azaïs and Wschebor [8]) even in the case the process has 1-dimensional parameter (Azaïs and Wschebor [7]). Let us remark that Theorem 3 holds true for non-Gaussian processes under appropriate conditions allowing to apply Rice formula.

Our method can be exploited to study higher order differentiability of F_M (as it has been done in [7] for one-parameter processes) but we will not pursue this subject here.

This paper is organized as follows:

Section 2 includes an extension of Rice Formula which gives an integral expression for the expectation of the weighted number of roots of a random system of d equations with d real unknowns. A complete proof of this formula in a form which is adapted to our needs in this paper, can be found in [9]. There is an extensive literature on Rice formula in various contexts (see for example Belayiev [10] , Cramér-Leadbetter [15], Marcus [23], Adler [1], Wschebor [35].

In Section 3, we obtain the exact expression for the distribution of the maximum as a consequence of the Rice-like formula of the previous section. This immediately implies the existence of the density and gives the implicit formula for it. The proof avoids unnecessary technicalities that we have used in previous work, even in cases that are much simpler than the ones considered here.

In Section 4, we compute (Theorem 4) the first order approximation in the direct method for stationary isotropic processes defined on a polyhedron, from which a new upper bound for $\mathrm{P}\{M > u\}$ for all real u follows.

In Section 5, we consider second order approximation, both for the direct method and the EPC approximation method. This is the content of Theorems 5, 6 and 7.

Section 6 contains some examples.

Assumptions and notations

$\mathcal{X} = \{X(t) : t \in S\}$ denotes a real-valued Gaussian field defined on the parameter set S. We assume that S satisfies the hypothesis A1

A1 :

- S is a compact subset of $\mathbb{R}^d$

- S is the disjoint union of $S_d, S_{d-1}..., S_0$, where S_j is an orientable C^3 manifold of dimension j without boundary. The S_j's will be called faces. Let S_{d_0}, $d_0 \leq d$ be the non empty face having largest dimension.

- We will assume that each S_j has an atlas such that the second derivatives of the inverse functions of all charts (viewed as diffeomorphisms from an open set in $\mathbb{R}^j$ to S_j) are bounded by a fixed constant. For $t \in S_j$, we denote L_t the maximum curvature of S_j at the point t. It follows that L_t is bounded for $t \in S$.

Notice that the decomposition $S = S_d \cup ... \cup S_0$ is not unique.

Concerning the random field we make the following assumptions A2-A5

A2 : $\mathcal{X}$ is in fact defined on an open set containing S and has $\mathcal{C}^2$ paths

A3 : for every $t \in S$ the distribution of $\big(X(t), X'(t)\big)$ does not degenerate; for every $s, t \in S$, $s \neq t$, the distribution of $\big(X(s), X(t)\big)$ does not degenerate.

A4 : Almost surely the maximum of $X(t)$ on S is attained at a single point.

For $t \in S_j$, $X'_j(t)$ $X'_{j,N}(t)$ denote respectively the derivative along S_j and the normal derivative. Both quantities are viewed as vectors in $\mathbb{R}^d$, and the density of their distribution will be expressed respectively with respect to an orthonormal basis of the tangent space $T_{t,j}$ of S_j at the point t, or its orthogonal complement $N_{t,j}$. $X''_j(t)$ will denote the second derivative of X along S_j, at the point $t \in S_j$ and will be viewed as a matrix expressed in an orthogonal basis of $T_{t,j}$. Similar notations will be used for any function defined on S_j.

A5 : Almost surely, for every $j = 0, 1, \dots, d$ there is no point t in S_j such that $X'_j(t) = 0$, $\det(X''_j(t)) = 0$

Other notations and conventions will be as follows :

- σ_j is the geometric measure on S_j.

- $m(t) := \mathrm{E}(X(t))$, $r(s,t) = \mathrm{Cov}(X(s), X(t))$ denote respectively the expectation and covariance of the process $\mathcal{X}$; $r_{0,1}(s,t)$, $r_{0,2}(s,t)$ are the first and the second derivatives of r with respect to t. Analogous notations will be used for other derivatives without further reference.

- If η is a random variable taking values in some Euclidean space, $p_\eta(x)$ will denote the density of its probability distribution with respect to the Lebesgue measure, whenever it exists.

- $\varphi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the standard Gaussian density ; $\Phi(x) := \int_{-\infty}^{x} \varphi(y)dy$.

- Assume that the random vectors ξ, η have a joint Gaussian distribution, where η has values in some finite dimensional Euclidean space. When it is well defined,

$$\mathrm{E}(f(\xi)/\eta = x)$$

  is the version of the conditional expectation obtained using Gaussian regression.

- $E_u := \{t \in S : X(t) > u\}$ is the excursion set above u of the function $X(.)$ and $A_u := \{M \leq u\}$ is the event that the maximum is not larger than u.

- $\langle,\rangle, \|\|$, denote respectively inner product and norm in a finite-dimensional real Euclidean space; λ_d is the Lebesgue measure on $\mathbb{R}^d$; $\mathcal{S}^{d-1}$ is the unit sphere ; A^c is the complement of the set A. If M is a real square matrix, $M \succ 0$ denotes that it is positive definite.

- If $g : D \to C$ is a function and $u \in C$, we denote

$$N_u^g(D) := \sharp\{t \in D : g(t) = u\}$$

which may be finite or infinite.

Some remarks on the hypotheses

One can give simple sufficient additional conditions on the process $\mathcal{X}$ so that A4 and A5 hold true.

If we assume that for each pair $j, k = 0, \dots, d$ and each pair of distinct points s, t, $s \in S_j, t \in S_k$, the distribution of the triplet

$$\big(X(t) - X(s), X_j'(s), X_k'(t)\big)$$

does not degenerate in $\mathbb{R} \times \mathbb{R}^j \times \mathbb{R}^k$, then A4 holds true.

This is well-known and follows easily from the next lemma (called Bulinskaya 's lemma) that we state without proof, for completeness.

Lemma 1 *Let $Z(t)$ be a stochastic process defined on some neighborhood of a set T embedded in some Euclidean space. Assume that the Hausdorff dimension of T is smaller or equal than the integer m and that the values of Z lie in $\mathbb{R}^{m+k}$ for some positive integer k . Suppose, in addition, that Z has $\mathcal{C}^1$ paths and that the density $p_{Z(t)}(v)$ is bounded for $t \in T$ and v in some neighborhood of $u \in \mathbb{R}^{m+k}$. Then, a. s. there is no point $t \in T$ such that $Z(t) = u$.*

With respect to A5, one has the following sufficient conditions: Assume A1, A2, A3 and as additional hypotheses one of the following two:

- $t \rightsquigarrow X(t)$ is of class $\mathcal{C}^3$

- $$\sup_{t \in S, x' \in V(0)} \mathrm{P}\big(|\det\big(X''(t)\big)| < \delta / X'(t) = x'\big) \to 0, \quad \text{as } \delta \to 0,$$

  where $V(0)$ is some neighborhood of zero.

Then A5 holds true. This follows from Proposition 2.1 of [8] and [16].

2 Rice formula for the number of weighted roots of random fields

In this section we review Rice formula for the expectation of the number of roots of a random system of equations. For proofs, see for example [8], or [9], where a simpler one is given.

Theorem 1 (Rice formula) *Let $Z : U \to \mathbb{R}^d$ be a random field, U an open subset of $\mathbb{R}^d$ and $u \in \mathbb{R}^d$ a fixed point in the codomain. Assume that:*
(i) Z is Gaussian,
(ii) almost surely the function $t \rightsquigarrow Z(t)$ is of class $\mathcal{C}^1$,
(iii) for each $t \in U$, $Z(t)$ has a non degenerate distribution (i.e. $\mathrm{Var}\big(Z(t)\big) \succ 0$),
(iv) $\mathrm{P}\{\exists t \in U, Z(t) = u, \det\big(Z'(t)\big) = 0\} = 0$

Then, for every Borel set B contained in U, one has

$$\mathrm{E}\left(N_u^Z(B)\right) = \int_B \mathrm{E}\left(|\det(Z'(t))| / Z(t) = u\right)\ p_{Z(t)}(u) dt. \tag{2}$$

If B is compact, then both sides in (2) are finite.

Theorem 2 *Let Z be a random field that verifies the hypotheses of Theorem 1. Assume that for each $t \in U$ one has another random field $Y^t : W \to \mathbb{R}^{d'}$, where W is some topological space, verifying the following conditions:*

a) $Y^t(w)$ is a measurable function of (ω, t, w) and almost surely, $(t, w) \rightsquigarrow Y^t(w)$ is continuous.

b) For each $t \in U$ the random process $(s, w) \rightsquigarrow \big(Z(s), Y^t(w)\big)$ defined on $U \times W$ is Gaussian.

Moreover, assume that $g : U \times \mathcal{C}(W, \mathbb{R}^{d'}) \to \mathbb{R}$ is a bounded function, which is continuous when one puts on $\mathcal{C}(W, \mathbb{R}^{d'})$ the topology of uniform convergence on compact sets. Then, for each compact subset I of U, one has

$$\mathrm{E}\Big(\sum_{t \in I, Z(t)=u} g(t, Y^t)\Big) = \int_I \mathrm{E}\big(|\det(Z'(t)|g(t, Y^t)/Z(t) = u\big).p_{Z(t)}(u)dt. \tag{3}$$

Remarks:

1. We have already mentioned in the previous section sufficient conditions implying hypothesis (iv) in Theorem 1.

2. With the hypotheses of Theorem 1 it follows easily that if J is a subset of U, $\lambda_d(J) = 0$, then $\mathrm{P}\{N_u^Z(J) = 0\} = 1$ for each $u \in \mathbb{R}^d$.

3 The implicit formula for the density of the maximum

Theorem 3 *Under assumptions A1 to A5, the distribution of M has the density*

$$\begin{aligned} p_M(x) = &\sum_{t \in S_0} \mathrm{E}\big(\mathbb{1}_{A_x}/X(t) = x\big)p_{X(t)}(x) \\ &+ \sum_{j=1}^{d} \int_{S_j} \mathrm{E}\big(|\det(X_j''(t))|\ \mathbb{1}_{A_x}/X(t) = x, X_j'(t) = 0\big)p_{X(t),X_j'(t)}(x, 0)\sigma_j(dt), \end{aligned} \tag{4}$$

Remark: One can replace $|\det(X_j''(t))|$ in the conditional expectation by $(-1)^j \det(X_j''(t))$, since under the conditioning and whenever $M \le x$ holds true, $X_j''(t)$ is negative semi-definite.

Proof of Theorem 3

Let $N_j(u), j = 0, \dots, d$ be the number of global maxima of $X(.)$ on S that belong to S_j and are larger than u. From the hypotheses it follows that a.s. $\sum_{j=0,\dots,d} N_j(u)$ is equal to 0 or 1, so that

$$\mathrm{P}\{M > u\} = \sum_{j=0,\dots,d} \mathrm{P}\{N_j(u) = 1\} = \sum_{j=0,\dots,d} \mathrm{E}(N_j(u)). \tag{5}$$

The proof will be finished as soon as we show that each term in (5) is the integral over $(u, +\infty)$ of the corresponding term in (4).

This is self-evident for $j = 0$. Let us consider the term $j = d$. We apply the weighted Rice formula of Section 2 as follows :

- Z is the random field X' defined on S_d.
- For each $t \in S_d$, put $W = S$ and $Y^t : S \to \mathbb{R}^2$ defined as:

$$Y^t(w) := \big(X(w) - X(t), X(t)\big).$$

Notice that the second coordinate in the definition of Y^t does not depend on w.

- In the place of the function g, we take for each $n = 1, 2, \dots$ the function g_n defined as follows:

$$g_n(t, f_1, f_2) = \ g_n(f_1, f_2) = \big(1 - \mathcal{F}_n(\sup_{w \in S} f_1(w))\big).\big(1 - \mathcal{F}_n(u - f_2(\overline{w}))\big),$$

  where $\overline{w}$ is any point in W and for n a positive integer and $x \geq 0$, we define :

$$\mathcal{F}_n(x) := \mathcal{F}(nx) \ \ ; \quad \text{with } \mathcal{F}(x) = 0 \text{ if } 0 \leq x \leq 1/2 \ \ , \ \ \mathcal{F}(x) = 1 \text{ if } x \geq 1 \ , \qquad (6)$$

  and $\mathcal{F}$ monotone non-decreasing and continuous.

It is easy to check that all the requirements in Theorem 2 are satisfied, so that, for the value 0 instead of u in formula (3) we get:

$$\mathrm{E}\big(\sum_{t \in S_d, X'(t)=0} g_n(Y^t)\big) = \int_{S_d} \mathrm{E}\big(|\det(X''(t)| g_n(Y^t)/X'(t) = 0).p_{X'(t)}(0)\lambda_d(dt). \qquad (7)$$

Notice that the formula holds true for each compact subset of S_d in the place of S_d, hence for S_d itself by monotone convergence.
Let now $n \to \infty$ in (7). Clearly $g_n(Y^t) \downarrow \ \mathbb{I}_{X(s)-X(t) \leq 0, \forall s \in S}.\ \mathbb{I}_{X(t) \geq u}$. The passage to the limit does not present any difficulty since $0 \leq g_n(Y^t) \leq 1$ and the sum in the left-hand side is bounded by the random variable $N_0^{X'}(\overline{S_d})$, which is in L^1 because of Rice Formula. We get

$$\mathrm{E}(N_d(u)) = \int_{S_d} \mathrm{E}\big(|\det(X''(t)| \ \mathbb{I}_{X(s)-X(t) \leq 0, \forall s \in S} \ \mathbb{I}_{X(t) \geq u}/X'(t) = 0).p_{X'(t)}(0)\lambda_d(dt)$$

Conditioning on the value of $X(t)$, we obtain the desired formula for $j = d$.

The proof for $1 \leq j \leq d-1$ is essentially the same, but one must take care of the parameterization of the manifold S_j. One can first establish locally the formula on a chart of S_j, using local coordinates.

It can be proved as in [8], Proposition 2.2 (the only modification is due to the term $\mathbb{I}_{A_x}$) that the quantity written in some chart as

$$\mathrm{E}\big(\det(Y''(s)) \ \mathbb{I}_{A_x}/Y(s) = x, Y'(s) = 0\big)p_{Y(s),Y_j'(s)}(x, 0)ds,$$

where the process $Y(s)$ is the process X written in some chart of S_j ,
$(Y(s) = X(\phi^{-1}(s)))$, defines a j-form. By a j-form we mean a mesure on S_j that does not depend on the parameterization and which has a density with respect to the Lebesgue measure ds in every chart. It can be proved also that the integral of this j-form on S_j gives the expectation of $N_j(u)$.

To get formula (2) it suffices to consider locally around a precise point $t \in S_j$ the chart ϕ given by the projection on the tangent space at t. In this case we obtain that at t

- ds is in fact $\sigma_j(dt)$
- $Y'(s)$ is isometric to $X_j'(t)$

where $s = \phi(t)$. □

The first consequence of Theorem 3 is the next corollary. For the statement, we need to introduce some further notations.

For t in S_j, $j \leq d_0$ we define $\mathcal{C}_{t,j}$ as the closed convex cone generated by the set of directions:

$$\{\lambda \in \mathbb{R}^d : \ \|\lambda\| = 1 \ ; \exists \ s_n \in S, (n = 1, 2, \dots) \text{ such that } s_n \to t, \frac{t - s_n}{\|t - s_n\|} \to \lambda \text{ as } n \to +\infty\},$$

whenever this set is non-empty and $\mathcal{C}_{t,j} = \{0\}$ if it is empty. We will denote by $\widehat{\mathcal{C}}_{t,j}$ the dual cone of $\mathcal{C}_{t,j}$, that is:

$$\widehat{\mathcal{C}}_{t,j} := \{z \in \mathbb{R}^d : \langle z, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathcal{C}_{t,j}\}.$$

Notice that these definitions easily imply that $T_{t,j} \subset C_{t,j}$ and $\widehat{\mathcal{C}}_{t,j} \subset N_{t,j}$. Remark also that for $j = d_0$, $\widehat{\mathcal{C}}_{t,j} = N_{t,j}$.

We will say that the function $X(.)$ has an "extended outward" derivative at the point t in S_j, $j \leq d_0$ if $X'_{j,N}(t) \in \widehat{\mathcal{C}}_{t,j}$.

Corollary 1 *Under assumptions A1 to A5, one has :*

(a) $p_M(x) \leq \overline{p}(x)$ *where*

$$\overline{p}(x) := \sum_{t \in S_0} \mathrm{E}\big(\mathbb{1}_{X'(t) \in \widehat{\mathcal{C}}_{t,0}} / X(t) = x \big) p_{X(t)}(x) +$$
$$\sum_{j=1}^{d_0} \int_{S_j} \mathrm{E}\big(|\det(X''_j(t))| \, \mathbb{1}_{X'_{j,N}(t) \in \widehat{\mathcal{C}}_{t,j}} / X(t) = x, X'_j(t) = 0 \big) p_{X(t), X'_j(t)}(x, 0) \sigma_j(dt). \quad (8)$$

(b) $\mathrm{P}\{M > u\} \leq \int_u^{+\infty} \overline{p}(x) dx$.

Proof

(a) follows from Theorem 3 and the observation that if $t \in S_j$, one has
$\{M \leq X(t)\} \subset \{X'_{j,N}(t) \in \widehat{\mathcal{C}}_{t,j}\}$. (b) is an obvious consequence of (a). $\square$

The actual interest of this Corollary depends on the feasibility of computing $\overline{p}(x)$. It turns out that it can be done in some relevant cases, as we will see in the remaining of this section. Our result can be compared with the approximation of $\mathrm{P}\{M > u\}$ by means of $\int_u^{+\infty} p^E(x) dx$ given by [3], [34] where

$$p^E(x) := \sum_{t \in S_0} \mathrm{E}\big(\mathbb{1}_{X'(t) \in \widehat{\mathcal{C}}_{t,0}} / X(t) = x \big) p_{X(t)}(x)$$
$$+ \sum_{j=1}^{d_0} (-1)^j \int_{S_j} \mathrm{E}\big(\det(X''_j(t)) \, \mathbb{1}_{X'_{j,N}(t) \in \widehat{\mathcal{C}}_{t,j}} / X(t) = x, X'_j(t) = 0 \big) p_{X(t), X'_j(t)}(x, 0) \sigma_j(dt). \quad (9)$$

Under certain conditions , $\int_u^{+\infty} p^E(x) dx$ is the expected value of the EPC of the excursion set E_u (see [3]). The advantage of $p^E(x)$ over $\overline{p}(x)$ is that one can have nice expressions for it in quite general situations. Conversely $\overline{p}(x)$ has the obvious advantage that it is an upper-bound of the true density $p_M(x)$ and hence provides upon integrating once, an upper-bound for the tail probability, **for every** u **value**. It is not known whether a similar inequality holds true for $p^E(x)$.

On the other hand, under additional conditions, both provide good first order approximations for $p_M(x)$ as $x \to \infty$ as we will see in the next section. In the special case in which the process $\mathcal{X}$ is centered and has a law that is invariant under isometries and translations, we describe below a procedure to compute $\overline{p}(x)$.

4 Computing $\overline{p}(x)$ for stationary isotropic Gaussian fields

For one-parameter centered Gaussian process having constant variance and satisfying certain regularity conditions, a general bound for $p_M(x)$ has been computed in [8], pp.75-77. In the two parameter case, Mercadier [26] has shown a bound for $\mathrm{P}\{M > u\}$, obtained by means of a method especially suited to dimension 2. When the parameter is one or two-dimensional, these bounds are sharper than the ones below which, on the other hand, apply to any dimension but to a more restricted context. We will assume now that the process $\mathcal{X}$ is centered Gaussian, with a covariance function that can be written as

$$\mathrm{E}\big(X(s).X(t)\big) = \rho\big(\|s-t\|^2\big), \tag{10}$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}$ is of class $\mathcal{C}^4$. Without loss of generality, we assume that $\rho(0) = 1$. Assumption (10) is equivalent to saying that the law of $\mathcal{X}$ is invariant under isometries (i.e. linear transformations that preserve the scalar product) and translations of the underlying parameter space $\mathbb{R}^d$.

We will also assume that the set S is a polyhedron. More precisely we assume that each $S_j (j = 1, \ldots, d)$ is a union of subsets of affine manifolds of dimension j in $\mathbb{R}^d$.

The next lemma contains some auxiliary computations which are elementary and left to the reader. We use the abridged notation : $\rho' := \rho'(0)$, $\rho'' := \rho''(0)$

Lemma 2 *Under the conditions above, for each $t \in U$, $i, i', k, k', j = 1, \ldots, d$:*

1. $\mathrm{E}\big(\frac{\partial X}{\partial t_i}(t).X(t)\big) = 0,$
2. $\mathrm{E}\big(\frac{\partial X}{\partial t_i}(t).\frac{\partial X}{\partial t_k}(t)\big) = -2\rho'\delta_{ik}$ *and* $\rho' < 0,$
3. $\mathrm{E}\big(\frac{\partial^2 X}{\partial t_i \partial t_k}(t).X(t)\big) = 2\rho'\delta_{ik}, \mathrm{E}\big(\frac{\partial^2 X}{\partial t_i \partial t_k}(t).\frac{\partial X}{\partial t_j}(t)\big) = 0$
4. $\mathrm{E}\big(\frac{\partial^2 X}{\partial t_i \partial t_k}(t).\frac{\partial^2 X}{\partial t_{i'} \partial t_{k'}}(t)\big) = 24\rho''\big[\delta_{ii'}.\delta_{kk'} + \delta_{i'k}.\delta_{ik'} + \delta_{ik}\delta_{i'k'}\big],$
5. $\rho'' - \rho'^2 \geq 0$
6. *If $t \in S_j$, the conditional distribution of $X_j''(t)$ given $X(t) = x, X_j'(t) = 0$ is the same as the unconditional distribution of the random matrix*

   $$Z + 2\rho' x I_j \ ,$$

   where $Z = (Z_{ik} : i, k = 1, \ldots, j)$ is a symmetric $j \times j$ matrix with centered Gaussian entries, independent of the pair $\big(X(t), X'(t)\big)$ such that, for $i \leq k$, $i' \leq k'$ one has :

   $$\mathrm{E}(Z_{ik}Z_{i'k'}) = 4\big[2\rho''\delta_{ii'} + (\rho'' - \rho'^2)\big]\delta_{ik}\delta_{i'k'} + 4\rho''\delta_{ii'}.\delta_{kk'}(1 - \delta_{ik}) \ .$$

Let us introduce some additional notations:

- $H_n(x), n = 0, 1, \ldots$ are the standard Hermite polynomials, i.e.

  $$H_n(x) := e^{x^2}\big(-\frac{\partial}{\partial x}\big)^n e^{-x^2}$$

  For the properties of the Hermite polynomials we refer to Mehta [25].

- $\overline{H}_n(x), n = 0, 1, \ldots$ are the modified Hermite polynomials, defined as:

  $$\overline{H}_n(x) := e^{x^2/2}\big(-\frac{\partial}{\partial x}\big)^n e^{-x^2/2}$$

We will use the following result:

Lemma 3 *Let*

$$J_n(x) := \int_{-\infty}^{+\infty} e^{-y^2/2} H_n(\nu) dy, \quad n = 0, 1, 2, \dots \tag{11}$$

where ν stands for the linear form $\nu = ay + bx$ where a, b are some real parameters that satisfy $a^2 + b^2 = 1/2$. Then

$$J_n(x) := (2b)^n \sqrt{2\pi}\ \overline{H}_n(x).$$

Proof :

It is clear that J_n is a polynomial having degree n. Differentiating in (11) under the integral sign, we get:

$$J_n'(x) = b \int_{-\infty}^{+\infty} e^{-y^2/2} H_n'(\nu) dy = 2nb \int_{-\infty}^{+\infty} e^{-y^2/2} H_{n-1}(\nu) dy = 2n\ b\ J_{n-1}(x) \tag{12}$$

Also:

$$J_n(0) = \int_{-\infty}^{+\infty} e^{-y^2/2} H_n(ay) dy,$$

so that $J_n(0) = 0$ if n is odd.
If n is even, $n \geq 2$, using the standard recurrence relations for Hermite polynomials, we have:

$$\begin{aligned} J_n(0) &= \int_{-\infty}^{+\infty} e^{-y^2/2} \big[2ay H_{n-1}(ay) - 2(n-1) H_{n-2}(ay)\big] dy \\ &= 2a^2 \int_{-\infty}^{+\infty} e^{-y^2/2} H_{n-1}'(ay) dy - 2(n-1) J_{n-2}(0) \\ &= -4b^2 (n-1) J_{n-2}(0). \end{aligned} \tag{13}$$

Equality (13) plus $J_0(x) = \sqrt{2\pi}$ for all $x \in \mathbb{R}$, imply that:

$$J_{2p}(0) = (-1)^p (2b)^{2p} (2p-1)!! \sqrt{2\pi} = (-2b^2)^p \frac{(2p)!}{p!}\ \sqrt{2\pi}. \tag{14}$$

Now we can go back to (12) and integrate successively for $n = 1, 2, \dots$ on the interval $[0, x]$ using the initial value given by (14) when $n = 2p$ and $J_n(0) = 0$ when n is odd, obtaining :

$$J_n(x) = (2b)^n \sqrt{2\pi} Q_n(x),$$

where the sequence of polynomials $Q_n, n = 0, 1, 2, \dots$ verifies the conditions:

$$Q_0(x) = 1 \tag{15}$$

$$Q_n'(x) = n Q_n(x) \tag{16}$$

$$Q_n(0) = 0 \ \text{ if } n \text{ is odd} \tag{17}$$

$$Q_n(0) = (-1)^{n/2} (n-1)!! \ \text{ if } n \text{ is even.} \tag{18}$$

It is now easy to show that in fact $Q_n(x) = \overline{H}_n(x)$, $n = 0, 1, 2, \dots$ using for example that:

$$\overline{H}_n(x) = 2^{n/2} H_n\Big(\frac{x}{\sqrt{2}}\Big).$$

□

The integrals

$$I_n(v) = \int_{v}^{+\infty} e^{-t^2/2} H_n(t) dt,$$

will appear in our computations. They are computed in the next Lemma, which can be proved easily, using the standard properties of Hermite polynomials.

Lemma 4 *(a)*

$$I_n(v) = \; 2e^{-v^2/2} \sum_{k=0}^{[\frac{n-1}{2}]} 2^k \; \frac{(n-1)!!}{(n-1-2k)!!} H_{n-1-2k}(v) \tag{19}$$

$$+ \; \mathbb{I}_{\{n \text{ even}\}} \; 2^{\frac{n}{2}} \; (n-1)!! \sqrt{2\pi} \; \overline{\Phi}(x) \tag{20}$$

(b)

$$I_n(-\infty) = \; \mathbb{I}_{\{n \text{ even}\}} 2^{\frac{n}{2}} \; (n-1)!! \; \sqrt{2\pi} \tag{21}$$

Theorem 4 *Assume that the process $\mathcal{X}$ is centered Gaussian, satisfies conditions A1-A5 with a covariance having the form (10) and verifying the regularity conditions of the beginning of this section. Moreover, let S be a polyhedron. Then, $\overline{p}(x)$ can be expressed by means of the following formula:*

$$\overline{p}(x) = \varphi(x) \left\{ \sum_{t \in S_0} \widehat{\sigma}_0(t) + \sum_{j=1}^{d_0} \left[\left(\frac{|\rho'|}{\pi}\right)^{j/2} \overline{H}_j(x) + R_j(x) \right] g_j \right\} \;, \tag{22}$$

where

- *g_j is a geometric parameter of the face S_j defined by*

$$g_j = \int_{S_j} \widehat{\sigma}_j(t) \sigma_j(dt), \tag{23}$$

  where $\widehat{\sigma}_j(t)$ is the normalized solid angle of the cone $\widehat{\mathcal{C}}_{t,j}$ in $N_{t,j}$, that is:

$$\widehat{\sigma}_j(t) = \frac{\sigma_{d-j-1}(\widehat{\mathcal{C}}_{t,j} \cap \mathcal{S}^{d-j-1})}{\sigma_{d-j-1}(\mathcal{S}^{d-j-1})} \text{ for } j = 0, \dots, d-1, \tag{24}$$

$$\widehat{\sigma}_d(t) = 1. \tag{25}$$

  Notice that for convex or other usual polyhedra $\widehat{\sigma}_j(t)$ is constant for $t \in S_j$, so that g_j is equal to this constant multiplied by the j-dimensional geometric measure of S_j.

- *For $j = 1, \dots d$,*

$$R_j(x) = \left(\frac{2\rho''}{\pi|\rho'|}\right)^{\frac{j}{2}} \frac{\Gamma((j+1)/2}{\pi} \int_{-\infty}^{+\infty} T_j(v) \exp\left(-\frac{y^2}{2}\right) \, dy \tag{26}$$

  where

$$v := -(2)^{-1/2}\left((1-\gamma^2)^{1/2} y - \gamma x\right) \quad \text{with} \quad \gamma := |\rho'|(\rho'')^{-1/2} \tag{27}$$

  and

$$T_j(v) := \left[\sum_{k=0}^{j-1} \frac{H_k^2(v)}{2^k k!}\right] e^{-v^2/2} \; - \; \frac{H_j(v)}{2^j (j-1)!} I_{j-1}(v). \tag{28}$$

  where I_n is given in the previous Lemma.

For the proof of the theorem, we need some ingredients from random matrices theory. Following Mehta [25], denote by $q_n(\nu)$ the density of eigenvalues of $n \times n$ GOE matrices at the point ν, that is, $q_n(\nu)d\nu$ is the probability of G_n having an eigenvalue in the interval $(\nu, \nu + d\nu)$. The random $n \times n$ real random matrix G_n is said to have the GOE distribution, if it is symmetric, with centered Gaussian entries $g_{ik}, i, k = 1, \dots, n$ satisfying $\mathrm{E}(g_{ii}^2) = 1$, $\mathrm{E}(g_{ik}^2) = 1/2$ if $i < k$

and the random variables: $\{g_{ik},\ 1 \le i \le k \le n\}$ are independent.
It is well known that:

$$\begin{aligned}
e^{\nu^2/2} q_n(\nu) &= e^{-\nu^2/2} \sum_{k=0}^{n-1} c_k^2 H_k^2(\nu) \\
&\quad + 1/2\ (n/2)^{1/2} c_{n-1} c_n H_{n-1}(\nu) \Big[\int_{-\infty}^{+\infty} e^{-y^2/2} H_n(y) dy - 2 \int_{\nu}^{+\infty} e^{-y^2/2} H_n(y) dy \Big] \\
&\quad + \ \mathbb{1}_{\{n \text{ odd }\}} \frac{H_{n-1}(\nu)}{\int_{-\infty}^{+\infty} e^{-y^2/2} H_{n-1}(y) dy},
\end{aligned} \tag{29}$$

where $c_k := (2^k k! \sqrt{\pi})^{-1/2}, k = 0, 1, \ldots,$ (see Mehta [25], ch. 7.)

In the proof of the theorem we will use the following remark due to Fyodorov [18] that we state as a Lemma

Lemma 5 *Let G_n be a GOE $n \times n$ matrix. Then, for $\nu \in \mathbb{R}$ one has:*

$$\mathrm{E}\big(|\det(G_n - \nu I_n)|\big) = 2^{3/2} \Gamma\big((n+3)/2\big) \exp(\nu^2/2) \frac{q_{n+1}(\nu)}{n+1}, \tag{30}$$

Proof:

Denote by $\nu_1, \ldots, \nu_n$ the eigenvalues of G_n. It is well-known (Mehta [25], Kendall et al. [19]) that the joint density f_n of the n-tuple of random variables $(\nu_1, \ldots, \nu_n)$ is given by the formula

$$f_n(\nu_1, \ldots, \nu_n) = c_n \exp\Big(-\frac{\sum_{i=1}^n \nu_i^2}{2}\Big) \prod_{1 \le i < k \le n} |\nu_k - \nu_i| \ , \quad \text{with } c_n := (2\pi)^{-n/2} (\Gamma(3/2))^n \Big(\prod_{i=1}^n \Gamma(1+i/2)\Big)^{-1}$$

Then,

$$\begin{aligned}
\mathrm{E}\big(|\det(G_n - \nu I_n)|\big) &= \mathrm{E}\big(\prod_{i=1}^n |\nu_i - \nu|\big) \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^n |\nu_i - \nu| c_n \exp\big(-\frac{\sum_{i=1}^n \nu_i^2}{2}\big) \prod_{1 \le i < k \le n} |\nu_k - \nu_i| \ d\nu_1, \ldots, d\nu_n \\
&= e^{\nu^2/2} \frac{c_n}{c_{n+1}} \int_{\mathbb{R}^n} f_{n+1}(\nu_1, \ldots, \nu_n, \nu) d\nu_1, \ldots, d\nu_n = e^{\nu^2/2} \frac{c_n}{c_{n+1}} \frac{q_{n+1}(\nu)}{n+1} \ .
\end{aligned}$$

The remainder is plain. □

Proof of Theorem 4:

We use the definition (8) given in Corollary 1 and the moment computations of Lemma 2 which imply that:

$$p_{X(t)}(x) = \varphi(x) \tag{31}$$

$$p_{X(t), X'_j(t)}(x, 0) = \varphi(x) (2\pi)^{-j/2} (-2\rho')^{-j/2} \tag{32}$$

$$X'(t) \text{ is independent of } X(t) \tag{33}$$

$$X'_{j,N}(t) \text{ is independent of } (X''_j(t), X(t), X'_j(t)). \tag{34}$$

Since the distribution of $X'(t)$ is centered Gaussian with variance $-2\rho' I_d$, it follows that :

$$\mathrm{E}(\ \mathbb{1}_{X'(t) \in \widehat{C}_{t,0}} / X(t) = x) = \widehat{\sigma}_0(t) \quad \text{if } t \in S_0,$$

and if $t \in S_j, j \geq 1$:

$$\begin{aligned}\mathrm{E}(|\det(X_j''(t))|\ \mathbb{I}_{X_{j,N}'(t)\in\widehat{\mathcal{C}}_{t,j}}/X(t)=x, X_j'(t)=0)\\ = \widehat{\sigma}_j(t)\ \mathrm{E}(|\det(X_j''(t))|/X(t)=x, X_j'(t)=0)\\ = \widehat{\sigma}_j(t)\ \mathrm{E}(|\det(Z+2\rho' x I_j)|).\end{aligned} \tag{35}$$

In the formula above, $\widehat{\sigma}_j(t)$ is the normalized solid angle defined in the statement of the theorem and the random $j \times j$ real matrix Z has the distribution of Lemma 2 .
A standard moment computations shows that Z has the same distribution as the random matrix:

$$\sqrt{8\rho''}G_j + 2\sqrt{\rho'' - \rho'^2}\xi I_j,$$

where G_j is a $j \times j$ GOE random matrix, ξ is standard normal in $\mathbb{R}$ and independent of G_j. So, for $j \geq 1$ one has

$$\mathrm{E}\big(|\det(Z+2\rho' x I_j)|\big) = (8\rho'')^{j/2}\int_{-\infty}^{+\infty}\mathrm{E}\big(|\det(G_j - \nu I_j)|\big)\varphi(y)dy,$$

where ν is given by (27).
For the conditional expectation in (8) use this last expression in (35) and (5). For the density in (8) use (32). Then Lemma 3 gives (22). □

Remarks on the theorem

- The "principal term" is

$$\varphi(x)\left\{\sum_{t\in S_0}\widehat{\sigma}_0(t) + \sum_{j=1}^{d_0}\big[\big(\frac{|\rho'|}{\pi}\big)^{j/2}\overline{H}_j(x)\big]g_j\right\}\ , \tag{36}$$

  which is the product of a standard Gaussian density times a polynomial with degree d_0. Integrating once, we get -in our special case- the formula for the expectation of the EPC of the excursion set as given by [3]

- The "complementary term" given by

$$\varphi(x)\sum_{j=1}^{d_0} R_j(x)g_j, \tag{37}$$

  can be computed by means of a formula, as it follows from the statement of the theorem above. These formulae will be in general quite unpleasant due to the complicated form of $T_j(v)$. However, for low dimensions they are simple. For example:

$$T_1(v) = \sqrt{2\pi}\big[\varphi(v) - v(1-\Phi(v))\big], \tag{38}$$

$$T_2(v) = 2\sqrt{2\pi}\varphi(v), \tag{39}$$

$$T_3(v) = \sqrt{\frac{\pi}{2}}\big[3(2v^2+1)\varphi(v) - (2v^2-3)v(1-\Phi(v))\big]. \tag{40}$$

- Second order asymptotics for $p_M(x)$ as $x \to +\infty$ will be mainly considered in the next section. However, we state already that the complementary term (37) is equivalent, as $x \to +\infty$, to

$$\varphi(x)\ g_{d_0} K_{d_0} x^{2d_0-4}\ e^{-\frac{1}{2}\frac{\gamma^2}{3-\gamma^2}x^2}, \tag{41}$$

  where the constant K_j, $j = 1, 2, ...$ is given by:

$$K_j = 2^{3j-2} \frac{\Gamma\left(\frac{j+1}{2}\right)}{\sqrt{\pi}(2\pi\gamma)^{j/2}(j-1)!} \rho''^{j/4} \left(\frac{\gamma}{3-\gamma^2}\right)^{2j-4}. \tag{42}$$

  We are not going to go through this calculation, which is elementary but requires some work. An outline of it is the following. Replace the Hermite polynomials in the expression for $T_j(v)$ given by (28) by the well-known expansion:

$$H_j(v) = j! \sum_{i=0}^{[j/2]} (-1)^i \frac{(2v)^{j-2i}}{i!(j-2i)!} \tag{43}$$

  and $I_{j-1}(v)$ by means of the formula in Lemma 4.

  Evaluating the term of highest degree in the polynomial part, this allows to prove that, as $v \to +\infty$, $T_j(v)$ is equivalent to

$$\frac{2^{j-1}}{\sqrt{\pi}(j-1)!}\ v^{2j-4} e^{-\frac{v^2}{2}}. \tag{44}$$

  Using now the definition of $R_j(x)$ and changing variables in the integral in (26), one gets for $R_j(x)$ the equivalent:

$$K_j x^{2j-4}\ e^{-\frac{1}{2}\frac{\gamma^2}{3-\gamma^2}x^2}. \tag{45}$$

  In particular, the equivalent of (37) is given by the highest order non-vanishing term in the sum.

- Consider now the case in which S is the sphere $\mathcal{S}^{d-1}$ and the process satisfies the same conditions as in the theorem. Even though the theorem can not be applied directly, it is possible to deal with this example to compute $\overline{p}(x)$, only performing some minor changes. In this case, only the term that corresponds to $j = d-1$ in (8) does not vanish, $\widehat{C}_{t,d-1} = N_{t,d-1}$, so that $\mathbb{1}_{X'_{d-1,N}(t) \in \widehat{C}_{t,d-1}} = 1$ for each $t \in \mathcal{S}^{d-1}$ and one can use invariance under rotations to obtain:

$$\overline{p}(x) = \varphi(x)\ \frac{\sigma_{d-1}\left(\mathcal{S}^{d-1}\right)}{(2\pi)^{(d-1)/2}} \mathrm{E}\left(\left|\det(Z + 2\rho' x I_{d-1}) + (2|\rho'|)^{1/2}\eta I_{d-1}\right|\right) \tag{46}$$

  where Z is a $(d-1) \times (d-1)$ centered Gaussian matrix with the covariance structure of Lemma 2 and η is a standard Gaussian real random variable, independent of Z. (46) follows from the fact that the normal derivative at each point is centered Gaussian with

variance $2|\rho'|$ and independent of the tangential derivative. So, we apply the previous computation, replacing x by $x + (2|\rho'|)^{-1/2}\ \eta$ and obtain the expression:

$$\overline{p}(x) = \varphi(x)\ \frac{2\pi^{d/2}}{\Gamma(d/2)}$$
$$\int_{-\infty}^{+\infty} \big[(\frac{|\rho'|}{\pi})^{(d-1)/2} \overline{H}_{d-1}(x + (2|\rho'|)^{-1/2} y) + R_{d-1}(x + (2|\rho'|)^{-1/2} y) \big] \varphi(y) dy. \quad (47)$$

5 Asymptotics as $x \to +\infty$

In this section we will consider the errors in the direct and the EPC methods for large values of the argument x. Theses errors are:

$$\overline{p}(x) - p_M(x) = \sum_{t \in S_0} \mathrm{E}\big(\ \mathbb{1}_{X'(t) \in \widehat{C}_{t,0}} .\ \mathbb{1}_{M>x} / X(t) = x \big) p_{X(t)}(x)$$
$$+ \sum_{j=1}^{d_0} \int_{S_j} \mathrm{E}\big(|\det(X''_j(t)|\ \mathbb{1}_{X'_{j,N}(t) \in \widehat{C}_{t,j}} .\ \mathbb{1}_{M>x} \big) / X(t) = x, X'_j(t) = 0 \big) p_{X(t), X'_j(t)}(x, 0) \sigma_j(dt). \quad (48)$$

$$p^E(x) - p_M(x) = \sum_{t \in S_0} \mathrm{E}\big(\ \mathbb{1}_{X'(t) \in \widehat{C}_{t,0}} .\ \mathbb{1}_{M>x} / X(t) = x \big) p_{X(t)}(x)$$
$$+ \sum_{j=1}^{d_0} (-1)^j \int_{S_j} \mathrm{E}\big(\det(X''_j(t)\ \mathbb{1}_{X'_{j,N}(t) \in \widehat{C}_{t,j}} .\ \mathbb{1}_{M>x} \big) / X(t) = x, X'_j(t) = 0 \big) p_{X(t), X'_j(t)}(x, 0) \sigma_j(dt). \quad (49)$$

It is clear that for every real x,

$$|p^E(x) - p_M(x)| \leq \overline{p}(x) - p_M(x)$$

so that the upper bounds for $\overline{p}(x) - p_M(x)$ will automatically be upper bounds for $|p^E(x) - p_M(x)|$. Moreover, as far as the authors know, no better bounds for $|p^E(x) - p_M(x)|$ than for $\overline{p}(x) - p_M(x)$ are known. It is an open question to determine if there exist situations in which $p^E(x)$ is better asymptotically than $\overline{p}(x)$.

Our next theorem gives sufficient conditions allowing to ensure that the error

$$\overline{p}(x) - p_M(x)$$

is bounded by a Gaussian density having strictly smaller variance than the maximum variance of the given process $\mathcal{X}$, which means that the error is super- exponentially smaller than $p_M(x)$ itself, as $x \to +\infty$. In this theorem, we assume that the maximum of the variance is not attained in $S \backslash S_{d_0}$. This excludes constant variance or some other stationary-like condition that will be addressed in Theorem 6. As far as the authors know, the result of Theorem 5 is new even for one-parameter processes defined on a compact interval.

For parameter dimension $d_0 > 1$, the only result of this type for non-constant variance processes of which the authors are aware is Theorem 3.3 of [34].

Theorem 5 *Assume that the process $\mathcal{X}$ satisfies conditions A1 -A5. With no loss of generality, we assume that* $\max_{t \in S} \mathrm{Var}(X(t)) = 1$. *In addition, we will assume that the set S_v of points $t \in S$ where the variance of $X(t)$ attains its maximal value is contained in $S_{d_0} (d_0 > 0)$ the non-empty face having largest dimension and that no point in S_v is a boundary point of $S \backslash S_{d_0}$. Then, there exist some positive constants C, δ such that for every $x > 0$.*

$$|p^E(x) - p_M(x)| \leq \overline{p}(x) - p_M(x) \leq C \varphi(x(1 + \delta)), \quad (50)$$

where $\varphi(.)$ is the standard normal density.

Proof :

Let W be an open neighborhood of the compact subset S_v of S such that $dist(W,(S\backslash S_{d_0}))>0$ where $dist$ denote the Euclidean distance in $\mathbb{R}^d$. For $t\in S_j\cap W^c$, the density

$$p_{X(t),X'_j(t)}(x,0)$$

can be written as the product of the density of $X'_j(t)$ at the point 0, times the conditional density of $X(t)$ at the point x given that $X'_j(t)=0$, which is Gaussian with some bounded expectation and a conditional variance which is smaller than the unconditional variance, hence, bounded by some constant smaller than 1. Since the conditional expectations in (48) are uniformly bounded by some constant, due to standard bounds on the moments of the Gaussian law, one can deduce that:

$$\overline{p}(x)-p_M(x)=\int_{W\cap S_{d_0}} \mathrm{E}\big(|\det(X''_{d_0}(t))|\ \mathbb{1}_{X'_{d_0,N}(t)\in\widehat{C}_{t,d_0}}\cdot \mathbb{1}_{M>x}/X(t)=x,X'_{d_0}(t)=0\big)$$
$$\cdot p_{X(t),X'_{d_0}(t)}(x,0)\sigma_{d_0}(dt)+O(\varphi((1+\delta_1)x)), \quad (51)$$

as $x\to+\infty$, for some $\delta_1>0$. Our following task is to choose W such that one can assure that the first term in the right hand-member of (51) has the same form as the second, with a possibly different constant δ_1.
To do this , for $s\in S$ and $t\in S_{d_0}$, let us write the Gaussian regression formula of $X(s)$ on the pair $(X(t),X'_{d_0}(t))$:

$$X(s)=a^t(s)X(t)+\langle b^t(s),X'_{d_0}(t)\rangle+\frac{\|t-s\|^2}{2}X^t(s). \quad (52)$$

where the regression coefficients $a^t(s),b^t(s)$ are respectively real-valued and $\mathbb{R}^{d_0}$-valued.
From now onwards, we will only be interested in those $t\in W$. In this case, since W does not contain boundary points of $S\backslash S_{d_0}$, it follows that

$$\widehat{C}_{t,d_0}=N_{t,d_0}\quad\text{and}\quad \mathbb{1}_{X'_{d_0,N}(t)\in\widehat{C}_{t,d_0}}=1.$$

Moreover, whenever $s\in S$ is close enough to t, necessarily, $s\in S_{d_0}$ and one can show that the Gaussian process $\{X^t(s):t\in W\cap S_{d_0},s\in S\}$ is bounded, in spite of the fact that its trajectories are not continuous at $s=t$. For each t, $\{X^t(s):s\in S\}$ is a "helix process", see [8] for a proof of boundedness.
On the other hand, conditionally on $X(t)=x,X'_{d_0}(t)=0$ the event $\{M>x\}$ can be written as

$$\{X^t(s)>\beta^t(s)\ x,\quad \text{for some s}\in S\}$$

where

$$\beta^t(s)=\frac{2(1-a^t(s))}{\|t-s\|^2}. \quad (53)$$

Our next goal is to prove that if one can choose W in such a way that

$$\inf\{\beta^t(s):t\in W\cap S_{d_0},s\in S,s\neq t\}>0, \quad (54)$$

then we are done. In fact, apply the Cauchy-Schwarz inequality to the conditional expectation in (51). Under the conditioning, the elements of $X''_{d_0}(t)$ are the sum of affine functions of x with bounded coefficients plus centered Gaussian variables with bounded variances, hence, the absolute value of the conditional expectation is bounded by an expression of the form

$$\big(Q(t,x)\big)^{1/2}\Big(\mathrm{P}\big(\sup_{s\in S\backslash\{t\}}\frac{X^t(s)}{\beta^t(s)}>x\big)\Big)^{1/2}, \quad (55)$$

where $Q(t,x)$ is a polynomial in x of degree $2d_0$ with bounded coefficients. For each $t \in W \cap S_{d_0}$, the second factor in (55) is bounded by

$$\Big(\mathrm{P}\Big(\sup \big\{ \frac{X^t(s)}{\beta^t(s)} : t \in W \cap S_{d_0}, s \in S, s \neq t \big\} > x \Big) \Big)^{1/2}.$$

Now, we apply to the bounded separable Gaussian process

$$\Big\{ \frac{X^t(s)}{\beta^t(s)} : t \in W \cap S_{d_0}, s \in S, s \neq t \Big\}$$

the classical Landau-Shepp-Fernique inequality [20], [17] which gives the bound

$$\mathrm{P}\Big(\sup \big\{ \frac{X^t(s)}{\beta^t(s)} : t \in W \cap S_{d_0}, s \in S, s \neq t \big\} > x \Big) \leq C_2 \exp(-\delta_2 x^2),$$

for some positive constants C_2, δ_2 and any $x > 0$. Also, the same argument above for the density $p_{X(t),X'_{d_0}(t)}(x,0)$ shows that it is bounded by a constant times the standard Gaussian density. To finish, it suffices to replace these bounds in the first term at the right-hand side of (51).

It remains to choose W for (54) to hold true. Consider the auxiliary process

$$Y(s) := \frac{X(s)}{\sqrt{r(s,s)}}, \quad s \in S. \tag{56}$$

Clearly, $\mathrm{Var}(Y(s)) = 1$ for all $s \in S$. We set

$$r^Y(s,s') := \mathrm{Cov}(Y(s), Y(s')) \quad , \quad s, s' \in S.$$

Let us assume that $t \in S_v$. Since the function $s \rightsquigarrow \mathrm{Var}(X(s))$ attains its maximum value at $s = t$, it follows that $X(t), X'_{d_0}(t)$ are independent, on differentiation under the expectation sign. This implies that in the regression formula (52) the coefficients are easily computed and $a^t(s) = r(s,t)$ which is strictly smaller than 1 if $s \neq t$, because of the non-degeneracy condition.

Then

$$\beta^t(s) = \frac{2(1 - r(s,t))}{\|t-s\|^2} \geq \frac{2(1 - r^Y(s,t))}{\|t-s\|^2}. \tag{57}$$

Since $r^Y(s,s) = 1$ for every $s \in S$, the Taylor expansion of $r^Y(s,t)$ as a function of s, around $s = t$ takes the form:

$$r^Y(s,t) = 1 + \langle s-t, r^Y_{20,d_0}(t,t)(s-t) \rangle + o(\|s-t\|^2), \tag{58}$$

where the notation is self-explanatory.

Also, using that $\mathrm{Var}(Y(s)) = 1$ for $s \in S$, we easily obtain:

$$-r^Y_{20,d_0,}(t,t) = \mathrm{Var}(Y'_{d_0}(t)) = \mathrm{Var}(X'_{d_0}(t)) \tag{59}$$

where the last equality follows by differentiation in (56) and putting $s = t$. (59) implies that $-r^Y_{20,d_0,}(t,t)$ is uniformly positive definite on $t \in S_v$, meaning that its minimum eigenvalue has a strictly positive lower bound. This, on account of (57) and (58), already shows that

$$\inf\{\beta^t(s) : t \in S_v, s \in S, s \neq t\} > 0, \tag{60}$$

The foregoing argument also shows that

$$\inf\{-\tau(a^t)''_{d_0}(t)\tau : t \in S_v, \tau \in \mathcal{S}^{d_0-1}, s \neq t\} > 0, \tag{61}$$

since whenever $t \in S_v$, one has $a^t(s) = r(s,t)$ so that

$$(a^t)''_{d_0}(t) = r_{20,d_0,}(t,t).$$

To end up, assume there is no neighborhood W of S_v satisfying (54). In that case using a compactness argument, one can find two convergent sequences $\{s_n\} \subset S$, $\{t_n\} \subset S_{d_0}$, $s_n \to s_0$, $t_n \to t_0 \in S_v$ such that

$$\beta^{t_n}(s_n) \to \ell \leq 0.$$

ℓ may be $-\infty$.
$t_0 \neq s_0$ is not possible, since it would imply

$$\ell = 2\frac{(1 - a^{t_0}(s_0))}{\|t_0 - s_0\|^2} = \beta^{t_0}(s_0),$$

which is strictly positive.
If $t_0 = s_0$, on differentiating in (52) with respect to s along S_{d_0} we get:

$$X'_{d_0}(s) = (a^t)'_{d_0}(s)X(t) + \langle (b^t)'_{d_0}(s), X'_{d_0}(t)\rangle + \frac{\partial_{d_0}}{\partial s}\frac{\|t-s\|^2}{2}X^t(s),$$

where $(a^t)'_{d_0}(s)$ is a column vector of size d_0 and $(b^t)'_{d_0}(s)$ is a $d_0 \times d_0$ matrix. Then, one must have $a^t(t) = 1$, $(a^t)'_{d_0}(t) = 0$. Thus

$$\beta^{t_n}(s_n) = -u_n^T(a^{t_0})''_{d_0}(t_0)u_n + o(1),$$

where $u_n := (s_n - t_n)/\|s_n - t_n\|$. Since $t_0 \in S_v$ we may apply (61) and the limit ℓ of $\beta^{t_n}(s_n)$ cannot be non-positive. $\square$

A straightforward application of Theorem 5 is the following

Corollary 2 *Under the hypotheses of Theorem 5, there exists positive constants C, δ such that, for every $u > 0$:*

$$0 \leq \left|\int_u^{+\infty} p^E(x)dx - \mathrm{P}(M > u)\right| \leq \int_u^{+\infty} \overline{p}(x)dx - \mathrm{P}(M > u) \leq C\mathrm{P}(\xi > u),$$

where ξ is a centered Gaussian variable with variance $1 - \delta$

The precise order of approximation of $\overline{p}(x) - p_M(x)$ or $p^E(x) - p_M(x)$ as $x \to +\infty$ remains in general an open problem, even if one only asks for the constants σ_d^2, σ_E^2 respectively which govern the second order asymptotic approximation and which are defined by means of

$$\frac{1}{\sigma_d^2} := \lim_{x\to+\infty} -2x^{-2}\log\left[\overline{p}(x) - p_M(x)\right] \tag{62}$$

and

$$\frac{1}{\sigma_E^2} := \lim_{x\to+\infty} -2x^{-2}\log\left|p^E(x) - p_M(x)\right| \tag{63}$$

whenever these limits exist. In general, we are unable to compute the limits (62) or (63) or even to prove that they actually exist or differ. Our more general results (as well as in [3], [34]) only contain lower-bounds for the liminf as $x \to +\infty$. This is already interesting since it gives some upper-bounds for the speed of approximation for $p_M(x)$ either by $\overline{p}(x)$ or $p^E(x)$. On the other hand, in Theorem 7 below, we are able to prove the existence of the limit and compute σ_d^2 for a relevant class of Gaussian processes.

For the next theorem we need an additional condition on the parameter set S. For S verifying $A1$ we define

$$\kappa(S) = \sup_{0 \le j \le d_0} \sup_{t \in S_j} \sup_{s \in S, s \neq t} \frac{dist\big((t-s), C_{t,j}\big)}{\|s-t\|^2} \tag{64}$$

where $dist$ is the Euclidean distance in $\mathbb{R}^d$.
One can show that $\kappa(S) < \infty$ in each one of the following classes of parameter sets S:
- S is convex, in which case $\kappa(S) = 0$.
- S is a C^3 manifold, with or without boundary.
- S verifies the following condition: For every $t \in S$ there exists an open neighborhood V of t in $\mathbb{R}^d$ and a $\mathcal{C}^3$ diffeomorphism $\psi : V \to B(0,r)$ (where $B(0,r)$ denotes the open ball in $\mathbb{R}^d$ centered at 0 and having radius r, $r > 0$) such that

$$\psi(V \cap S) \;=\; C \cap B(0,r), \text{ where } C \text{ is a convex cone.}$$

However, $\kappa(S) < \infty$ can fail in general. A simple example showing what is going on is the following: take an orthonormal basis of $\mathbb{R}^2$ and put

$$S = \{(\lambda, 0) : 0 \le \lambda \le 1\} \cup \{(\mu \cos\theta, \mu \sin\theta) : 0 \le \mu \le 1\}$$

where $0 < \theta < \pi$, that is, S is the boundary of an angle of size θ. One easily checks that $\kappa(S) = +\infty$. Moreover it is known [3] that in this case the EPC approximation does not verify a super- exponential inequality. More generally, sets S having "whiskers" have $\kappa(S) = +\infty$.

Theorem 6 *Let $\mathcal{X}$ be a stochastic process on S satisfying A1 -A5. Suppose in addition that* $\mathrm{Var}(X(t)) = 1$ *for all* $t \in S$ *and that* $\kappa(S) < +\infty$.
Then

$$\liminf_{x \to +\infty} -2x^{-2} \log\left[\overline{p}(x) - p_M(x)\right] \ge 1 + \inf_{t \in S} \frac{1}{\sigma_t^2 + \overline{\lambda}(t)\kappa_t^2} \tag{65}$$

with

$$\sigma_t^2 := \sup_{s \in S \setminus \{t\}} \frac{\mathrm{Var}\big(X(s)/X(t), X'(t)\big)}{(1 - r(s,t))^2}$$

and

$$\kappa_t := \sup_{s \in S \setminus \{t\}} \frac{dist\Big(-\Lambda_t^{-1} r_{01}(s,t), \mathcal{C}_{t,j}\Big)}{1 - r(s,t)}, \tag{66}$$

where

- $\Lambda_t := \mathrm{Var}(X'(t))$
- $\overline{\lambda}(t)$ *is the maximum eigenvalue of* Λ_t
- *in (66), j is such that* $t \in S_j$,$(j = 0, 1, \ldots, d_0)$.

The quantity in the right hand side of (65) is strictly bigger than 1.

Remark. In formula (65) it may happen that the denominator in the right-hand side is identically zero, in which case we put $+\infty$ for the infimum. This is the case of the one-parameter process $X(t) = \xi \cos t + \eta \sin t$ where ξ, η are Gaussian standard independent random variables, and S is an interval having length strictly smaller than π.

Proof of Theorem 6

Let us first prove that $\sup_{t\in S}\kappa_t<\infty$.
For each $t\in S$, let us write the Taylor expansions

$$\begin{aligned} r_{01}(s,t) &= r_{01}(t,t)+r_{11}(t,t)(s-t)+O(\|s-t\|^2)\\ &= \Lambda_t(s-t)+O(\|s-t\|^2)\end{aligned}$$

where O is uniform on $s,t\in S$, and

$$1-r(s,t)=(s-t)^T\Lambda_t(s-t)+O(\|s-t\|^2)\geq L_2\|s-t\|^2,$$

where L_2 is some positive constant. It follows that for $s\in S,\ t\in S_j,\ s\neq t$, one has:

$$\frac{dist\Big(-\Lambda_t^{-1}r_{01}(s,t),\mathcal{C}_{t,j}\Big)}{1-r(s,t)}\ \leq\ L_3\frac{dist\big((t-s),\mathcal{C}_{t,j}\big)}{\|s-t\|^2}\ +\ L_4, \tag{67}$$

where L_3 and L_4 are positive constants. So,

$$\frac{dist\Big(-\Lambda_t^{-1}r_{01}(s,t),\mathcal{C}_{t,j}\Big)}{1-r(s,t)}\ \leq\ L_3\ \kappa(S)\ +\ L_4.$$

which implies $\sup_{t\in S}\kappa_t<\infty$.

With the same notations as in the proof of Theorem 5, using (4) and (8), one has:

$$\begin{aligned}\overline{p}(x)-p_M(x)=\varphi(x)\bigg[&\sum_{t\in S_0}\mathrm{E}\big(\ \mathbb{1}_{X_t'(t)\in\widehat{\mathcal{C}}_{t,0}}.\ \mathbb{1}_{M>x}/X(t)=x\big)\\ &+\sum_{j=1}^{d_0}\int_{S_j}\mathrm{E}\big(|\det(X_j''(t))|\ \mathbb{1}_{X_{j,N}'(t)\in\widehat{\mathcal{C}}_{t,j}}.\ \mathbb{1}_{M>x}/X(t)=x,X_j'(t)=0\big)\\ &\qquad(2\pi)^{-j/2}(\det(\mathrm{Var}(X_j'(t))))^{-1/2}\sigma_j(dt)\bigg].\end{aligned} \tag{68}$$

Proceeding in a similar way to that of the proof of Theorem 5, an application of the Hölder inequality to the conditional expectation in each term in the right-hand side of (68) shows that the desired result will follow as soon as we prove that:

$$\liminf_{x\to+\infty}-2x^{-2}\log\mathrm{P}\big(\{X_{j,N}'\in\widehat{\mathcal{C}}_{t,j}\}\cap\{M>x\}/X(t)=x,X_j'(t)=0\big)\geq\frac{1}{\sigma_t^2+\overline{\lambda}(t)\kappa_t^2}, \tag{69}$$

for each $j=0,1,\dots,d_0$, where the liminf has some uniformity in t.

Let us write the Gaussian regression of $X(s)$ on the pair $(X(t),X'(t))$

$$X(s)=a^t(s)X(t)+\langle b^t(s),X'(t)\rangle+R^t(s).$$

Since $X(t)$ and $X'(t)$ are independent, one easily computes :

$$\begin{aligned}a^t(s)&=r(s,t)\\ b^t(s)&=\Lambda_t^{-1}r_{01}(s,t).\end{aligned}$$

Hence, conditionally on $X(t)=x,\ X_j'(t)=0$, the events

$$\{M>x\}\ \text{ and }\ \{R^t(s)>(1-r(s,t))x-r_{01}^T(s,t)\Lambda_t^{-1}X_{j,N}'(t)\text{ for some }s\in S\}$$

coincide.
Denote by $(X'_{j,N}(t)|X'_j(t)=0)$ the regression of $X'_{j,N}(t)$ on $X'_j(t)=0$. So, the probability in (69) can written as

$$\int_{\widehat{\mathcal{C}}_{t,j}} \mathrm{P}\{\zeta^t(s) > x - \frac{r_{01}^T(s,t)\Lambda_t^{-1}x'}{1-r(s,t)} \text{ for some } s \in S\} p_{X'_{j,N}(t)|X'_j(t)=0}(x')dx' \tag{70}$$

where

- $\zeta^t(s) := \dfrac{R^t(s)}{1-r(s,t)}$
- dx' is the Lebesgue measure on $N_{t,j}$. Remember that $\widehat{\mathcal{C}}_{t,j} \subset N_{t,j}$.

If $-\Lambda_t^{-1}r_{01}(s,t) \in \mathcal{C}_{t,j}$ one has

$$-r_{01}^T(s,t)\Lambda_t^{-1}x' \geq 0$$

for every $x' \in \widehat{\mathcal{C}}_{t,j}$, because of the definition of $\widehat{\mathcal{C}}_{t,j}$.
If $-\Lambda_t^{-1}r_{01}(s,t) \notin \mathcal{C}_{t,j}$, since $\mathcal{C}_{t,j}$ is a closed convex cone, we can write

$$-\Lambda_t^{-1}r_{01}(s,t) = z' + z''$$

with $z' \in \mathcal{C}_{t,j}$, $z' \perp z''$ and $\|z''\| = dist(-\Lambda_t^{-1}r_{01}(s,t), \mathcal{C}_{t,j})$.
So, if $x' \in \widehat{\mathcal{C}}_{t,j}$:

$$\frac{-r_{01}^T(s,t)\Lambda_t^{-1}x'}{1-r(s,t)} = \frac{z'^T x' + z''^T x'}{1-r(s,t)} \geq -\kappa_t\|x'\|$$

using that $z'^T x' \geq 0$ and the Cauchy-Schwarz inequality. It follows that in any case, if $x' \in \widehat{\mathcal{C}}_{t,j}$ the expression in (70) is bounded by

$$\int_{\widehat{\mathcal{C}}_{t,j}} \mathrm{P}\Big(\zeta^t(s) > x - \kappa_t\|x'\| \text{ for some } s \in S\Big) p_{X'_{j,N}(t)|X'_j(t)=0}(x')dx'. \tag{71}$$

To obtain a bound for the probability in the integrand of (71) we will use the classical inequality for the tail of the distribution of the supremum of a Gaussian process with bounded paths.

The Gaussian process $(s,t)) \rightsquigarrow \zeta^t(s)$, defined on $(S \times S)\backslash\{s=t\}$ has continuous paths. As the pair (s,t) approches the diagonal of $S \times S$, $\zeta^t(s)$ may not have a limit but, almost surely, it is bounded (see [8] for a proof). (For fixed t, $\zeta^t(.)$ is a "helix process" with a singularity at $s=t$, a class of processes that we have already met above).

We set

- $m^t(s) := \mathrm{E}(\zeta^t(s)) \quad (s \neq t)$
- $m := \sup_{s,t\in S, s\neq t} |m^t(s)|$
- $\mu := \mathrm{E}\big(|\sup_{s,t\in S,s\neq t}\big[\zeta^t(s) - m^t(s)\big]|\big)$.

The almost sure boundedness of the paths of $\zeta^t(s)$ implies that $m < \infty$ and $\mu < \infty$. Applying the Borell-Sudakov-Tsirelson type inequality (see for example Adler [2] and references therein) to the centered process $s \rightsquigarrow \zeta^t(s) - m^t(s)$ defined on $S\backslash\{t\}$, we get whenever $x - \kappa_t\|x'\| - m - \mu > 0$:

$$\begin{aligned}\mathrm{P}\{\zeta^t(s) > x - \kappa_t\|x'\| \text{ for some } s \in S\} \\ &\leq \mathrm{P}\{\zeta^t(s) - m^t(s) > x - \kappa_t\|x'\| - m \text{ for some } s \in S\} \\ &\leq 2\exp\Big(-\frac{(x-\kappa_t\|x'\|-m-\mu)^2}{2\sigma_t^2}\Big).\end{aligned}$$

The Gaussian density in the integrand of (71) is bounded by

$$(2\pi\underline{\lambda}_j(t))^{\frac{j-d}{2}} \exp \frac{\|x' - m'_{j,N}(t)\|^2}{2\overline{\lambda}_j(t)}$$

where $\underline{\lambda}_j(t)$ and $\overline{\lambda}_j(t)$ are respectively the minimum and maximum eigenvalue of $\mathrm{Var}(X'_{j,N}(t)|X'_j(t))$ and $m'_{j,N}(t)$ is the conditional expectation $\mathrm{E}(X'_{j,N}(t)|X'_j(t)=0)$. Notice that $\underline{\lambda}_j(t), \overline{\lambda}_j(t), m'_{j,N}(t)$ are bounded, $\underline{\lambda}_j(t)$ is bounded below by a positive constant and $\overline{\lambda}_j(t) \leq \overline{\lambda}(t)$.

Replacing into (71) we have the bound :

$$\begin{aligned}
&\mathrm{P}\big(\{X'_{j,N} \in \widehat{\mathcal{C}}_{t,j}\} \cap \{M > x\}/X(t) = x, X'_j(t) = 0\big) \\
&\leq (2\pi\underline{\lambda}_j(t))^{\frac{j-d}{2}} 2 \int_{\widehat{\mathcal{C}}_{t,j}\cap\{x-\kappa_t\|x'\|-m-\mu>0\}} \exp -\Big(\frac{(x-\kappa_t\|x'\|-m-\mu)^2}{2\sigma_t^2} + \frac{\|x'-m'_{j,N}(t)\|^2}{2\overline{\lambda}(t)}\Big)\, dx' \\
&\qquad\qquad + \mathrm{P}\Big(\|X'_{j,N}(t)|X'_j(t)=0\| \geq \frac{x-m-\mu}{\kappa_t}\Big), \qquad (72)
\end{aligned}$$

where it is understood that the second term in the right-hand side vanishes if $\kappa_t = 0$.
Let us consider the first term in the right-hand side of (72). We have:

$$\begin{aligned}
&\frac{(x-\kappa_t\|x'\|-m-\mu)^2}{2\sigma_t^2} + \frac{\|x'-m'_{j,N}(t)\|^2}{2\overline{\lambda}(t)} \\
&\qquad \geq \frac{(x-\kappa_t\|x'\|-m-\mu)^2}{2\sigma_t^2} + \frac{(\|x'\|-\|m'_{j,N}(t)\|)^2}{2\overline{\lambda}(t)} \\
&\qquad = \big[A(t)\|x'\| + B(t)(x-m-\mu) + C(t)\big]^2 + \frac{(x-m-\mu-\kappa_t\|m'_{j,N}(t)\|)^2}{2\sigma_t^2 + 2\overline{\lambda}(t)\kappa_t^2},
\end{aligned}$$

where the last inequality is obtained after some algebra, $A(t), B(t), C(t)$ are bounded functions and $A(t)$ is bounded below by some positive constant.

So the first term in the right-hand side of (72) is bounded by :

$$\begin{aligned}
&2.(2\pi\underline{\lambda}_j)^{\frac{j-d}{2}} \exp -\Big(\frac{(x-m-\mu-\kappa_t\|m'_{j,N}(t))^2}{2\sigma_t^2 + 2\overline{\lambda}(t)\kappa_t^2}\Big) \\
&\qquad \int_{R^{d-j}} \exp -\big[\big(A(t)\|x'\| + B(t)(x-m-\mu) + C(t)\big)\big]^2 dx' \\
&\qquad\qquad \leq L|x|^{d-j-1} \exp -\Big(\frac{(x-m-\mu-\kappa_t\|m'_{j,N}(t)\|)^2}{2\sigma_t^2 + 2\overline{\lambda}(t)\kappa_t^2}\Big) \qquad (73)
\end{aligned}$$

where L is some constant. The last inequality follows easily using polar coordinates.

Consider now the second term in the right-hand side of (72). Using the form of the conditional density $p_{X'_{j,N}(t)/X'_j(t)=0}(x')$, it follows that it is bounded by

$$\begin{aligned}
&\mathrm{P}\Big\{\|(X'_{j,N}(t)/X'_j(t)=0) - m'_{j,N}(t)\| \geq \frac{x-m-\mu-\kappa_t\|m'_{j,N}(t)\|}{\kappa_t}\Big\} \\
&\qquad\qquad \leq L_1|x|^{d-j-2} \exp -\Big(\frac{(x-m-\mu-\kappa_t\|m'_{j,N}(t)\|)^2}{2\overline{\lambda}(t)\kappa_t^2}\Big) \qquad (74)
\end{aligned}$$

where L_1 is some constant. Putting together (73) and (74) with (72), we obtain (69). □

The following two corollaries are straightforward consequences of Theorem 6:

Corollary 3 *Under the hypotheses of Theorem 6 one has*

$$\liminf_{x\to+\infty} -2x^{-2}\log|p^E(x)-p_M(x)| \geq 1+\inf_{t\in S}\frac{1}{\sigma_t^2+\overline{\lambda}(t)\kappa_t^2}.$$

Corollary 4 *Let $\mathcal{X}$ a stochastic process on S satisfying A1 -A5. Suppose in addition that* $\mathrm{E}(X(t))=0$, $\mathrm{E}(X^2(t))=1$, $\mathrm{Var}(X'(t)=I_d$ *for all* $t\in S$.
Then

$$\liminf_{u\to+\infty} -\ 2u^{-2}\log\Big|\mathrm{P}(M>u)-\int_u^{+\infty}p^E(x)dx\Big| \geq 1+\inf_{t\in S}\frac{1}{\sigma_t^2+\kappa_t^2}.$$

and

$$p^E(x)=\Big[\sum_{j=0}^{d_0}(-1)^j(2\pi)^{-j/2}g_j\overline{H}_j(x)\Big]\varphi(x).$$

where g_j is given by (23) and $\overline{H}_j(x)$ has been defined in Section 4.

The proof follows directly from Theorem 6 the definition of $p^E(x)$ and the results in [1].

6 Examples

1) A simple application of Theorem 5 is the following. Let $\mathcal{X}$ be a one parameter real-valued centered Gaussian process with regular paths, defined on the interval $[0,T]$ and satisfying an adequate non-degeneracy condition. Assume that the variance $v(t)$ has a unique maximum, say 1 at the interior point t_0, and $k=\min\{j : v^{(2j)}(t_0)\neq 0\}<\infty$. Notice that $v^{(2k)}(t_0)<0$. Then, one can obtain the equivalent of $p_M(x)$ as $x\to\infty$ which is given by:

$$p_M(x)\simeq\frac{1-v''(t_0)/2}{kC_k^{1/k}}\mathrm{E}\left(|\xi|^{\frac{1}{2k}-1}\right)x^{1-1/k}\varphi(x), \tag{75}$$

where ξ is a standard normal random variable and $C_k=-\frac{1}{(2k)!}v^{(2k)}(t_0)+\frac{1}{4}[v''(t_0)]^2\ \mathbb{1}_{k=2}$. The proof is a direct application of the Laplace method. The result is new for the density of the maximum, but if we integrate the density from u to $+\infty$, the corresponding bound for $\mathrm{P}\{M>u\}$ is known under weaker hypotheses (Piterbarg [28]).

2) Let the process $\mathcal{X}$ be centered and satisfy A1-A5. Assume that the the law of the process is isotropic and stationary, so that the covariance has the form (10) and verifies the regularity condition of Section 4. We add the simple normalization $\rho'=\rho'(0)=-1/2$. One can easily check that

$$\sigma_t^2=\sup_{s\in S\backslash\{t\}}\frac{1-\rho^2(\|s-t\|^2)-4\rho'^2(\|s-t\|^2)\|s-t\|^2}{[1-\rho(\|s-t\|^2)]^2} \tag{76}$$

Furthermore if

$$\rho'(x)\leq 0 \text{ for } x\geq 0 \tag{77}$$

one can show that the sup in (76) is attained as $\|s-t\|\to 0$ and is independent of t. Its value is

$$\sigma_t^2=12\rho''-1.$$

The proof is elementary (see [4] or [34]).

Let S be a convex set. For t $\in S_j$, $s\in S$:

$$dist\big(-r_{01}(s,t),\mathcal{C}_{t,j}\big)=dist\big(-2\rho'(\|s-t\|^2)(t-s),\mathcal{C}_{t,j}\big). \tag{78}$$

The convexity of S implies that $(t-s) \in \mathcal{C}_{t,j}$. Since $\mathcal{C}_{t,j}$ is a convex cone and $-2\rho'(\|s-t\|^2) \geq 0$, one can conclude that $-r_{01}(s,t) \in \mathcal{C}_{t,j}$ so that the distance in (78) is equal to zero. Hence,

$$\kappa_t = 0 \text{ for every } t \in S$$

and an application of Theorem 6 gives the inequality

$$\liminf_{x\to+\infty} -\frac{2}{x^2} \log\left[\overline{p}(x) - p_M(x)\right] \geq 1 + \frac{1}{12\rho'' - 1}. \tag{79}$$

A direct consequence is that the same inequality holds true when replacing $\overline{p}(x) - p_M(x)$ by $|p^E(x) - p_M(x)|$ in (79), thus obtainig the main explicit example in Adler and Taylor [3], or in Taylor et al. [34].

Next, we improve (79). In fact, under the same hypotheses, we prove that the liminf is an ordinary limit and the sign $\geq$ is an equality sign. We state this as

Theorem 7 *Assume that $\mathcal{X}$ is centered, satisfies hypotheses A1-A5, the covariance has the form (10) with $\rho'(0) = -1/2$, $\rho'(x) \leq 0$ for $x \geq 0$. Let S be a convex set, and $d_0 = d \geq 1$. Then*

$$\lim_{x\to+\infty} -\frac{2}{x^2} \log\left[\overline{p}(x) - p_M(x)\right] = 1 + \frac{1}{12\rho'' - 1}. \tag{80}$$

Remark Notice that since S is convex, the added hypothesis that the maximum dimension d_0 such that S_j is not empty is equal to d is not an actual restriction.

Proof of Theorem 7

In view of (79), it suffices to prove that

$$\limsup_{x\to+\infty} -\frac{2}{x^2} \log\left[\overline{p}(x) - p_M(x)\right] \leq 1 + \frac{1}{12\rho'' - 1}. \tag{81}$$

Using (4) and the definition of $\overline{p}(x)$ given by (8), one has the inequality

$$\overline{p}(x) - p_M(x) \geq (2\pi)^{-d/2}\varphi(x) \int_{S_d} \mathrm{E}\big(|\det(X''(t))|\ \mathbb{1}_{M>x}/X(t) = x, X'(t) = 0\big)\sigma_d(dt), \tag{82}$$

where our lower bound only contains the term corresponding to the largest dimension and we have already replaced the density $p_{X(t),X'(t)}(x,0)$ by its explicit expression using the law of the process. Under the condition $\{X(t) = x, X'(t) = 0\}$ if $v_0^T X''(t) v_0 > 0$ for some $v_0 \in \mathcal{S}^{d-1}$, a Taylor expansion implies that $M > x$. It follows that

$$\begin{aligned}\mathrm{E}\big(|\det(X''(t))|\ \mathbb{1}_{M>x}/X(t) = x, X'(t) = 0\big)&\\ \geq \mathrm{E}\big(|\det(X''(t))|\ \mathbb{1}_{\sup_{v\in\mathcal{S}^{d-1}} v^T X''(t) v > 0}/X(t) = x, X'(t) = 0\big).&\end{aligned} \tag{83}$$

We now apply Lemma 2 which describes the conditional distribution of $X''(t)$ given $X(t) = x, X'(t) = 0$. Using the notations of this lemma, we may write the right-hand side of (83) as :

$$\mathrm{E}\big(|\det(Z - xId)|\ \mathbb{1}_{\sup_{v\in\mathcal{S}^{d-1}} v^T Z v > x}\big),$$

which is obviously bounded below by

$$\begin{aligned}\mathrm{E}\big(|\det(Z - xId)|\ \mathbb{1}_{Z_{11}>x}\big)&\\ = \int_x^{+\infty} \mathrm{E}\big(|\det(Z - xId)|/Z_{11} = y\big)(2\pi)^{-1/2}\sigma^{-1}\exp\big(-\frac{y^2}{2\sigma^2}\big)dy,&\end{aligned} \tag{84}$$

where $\sigma^2 := \text{Var}(Z_{11}) = 12\rho'' - 1$. The conditional distribution of Z given $Z_{11} = y$ is easily deduced from Lemma 2. It can be represented by the random $d \times d$ real symmetric matrix

$$\widetilde{Z} := \begin{pmatrix} y & Z_{12} & \dots & \dots & Z_{1d} \\ & \xi_2 + \alpha y & Z_{23} & \dots & Z_{2d} \\ & & \ddots & & \\ & & & & \xi_d + \alpha y \end{pmatrix},$$

where the random variables $\{\xi_2, \dots, \xi_d, Z_{ik}, 1 \le i < k \le d\}$ are independent centered Gaussian with

$$\text{Var}(Z_{ik}) = 4\rho'' \quad (1 \le i < k \le d) \quad ; \quad \text{Var}(\xi_i) = \frac{16\rho''(8\rho'' - 1)}{12\rho'' - 1} \quad (i = 2, \dots, d) \quad ; \quad \alpha = \frac{4\rho'' - 1}{12\rho'' - 1}$$

Observe that $0 < \alpha < 1$.
Choose now α_0 such that $(1+\alpha_0)\alpha < 1$. The expansion of $\det(\widetilde{Z} - xId)$ shows that if $x(1+\alpha_0) \le y \le x(1+\alpha_0) + 1$ and x is large enough, then

$$\text{E}\big(|\det(\widetilde{Z} - xId)|\big) \ge L\ \alpha_0(1 - \alpha(1+\alpha_0))^{d-1}\ x^d,$$

where L is some positive constant. This implies that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_x^{+\infty} \exp(-\frac{y^2}{2\sigma^2})\text{E}\big(|\det(\widetilde{Z} - xId)|\big)dy \ge \frac{L}{\sqrt{2\pi}\sigma}\int_{x(1+\alpha_0)}^{x(1+\alpha_0)+1} \exp(-\frac{y^2}{2\sigma^2})\alpha_0(1-\alpha(1+\alpha_0))^{d-1}\ x^d dy$$

for x large enough. On account of (82),(83),(84), we conclude that for x large enough,

$$\overline{p}(x) - p_M(x) \ge L_1 x^d \exp -\Big[\frac{x^2}{2} + \frac{(x(1+\alpha_0)+1)^2}{2\sigma^2}\Big].$$

for some new positive constant L_1. Since α_0 can be chosen arbitrarily small, this implies (81).
□

3) Consider the same processes of Example 2, but now defined on the non-convex set $\{a \le \|t\| \le b\}$, $0 < a < b$. The same calculations as above show that $\kappa_t = 0$ if $a < \|t\| \le b$ and

$$\kappa_t = \max\Big\{\sup_{z\in[2a,a+b]} \frac{-2\rho'(z^2)z}{1-\rho(z^2)}, \sup_{\theta\in[0,\pi]} \frac{-2a\rho'(2a^2(1-\cos\theta))(1-cos\theta)}{1-\rho(2a^2(1-\cos\theta))}\Big\},$$

for $\|t\| = a$.

4) Let us keep the same hypotheses as in Example 2 but without assuming that the covariance is decreasing as in (77). The variance is still given by (76) but κ_t is not necessarily equal to zero. More precisely, relation (78) shows that

$$\kappa_t \le \sup_{s\in S\setminus\{t\}} 2\frac{\rho'(\|s-t\|^2)^+\|s-t\|}{1-\rho(\|s-t\|^2)}$$

The normalization: $\rho' = -1/2$ implies that the process $\mathcal{X}$ is "identity speed", that is $\text{Var}(X'(t)) = I_d$ so that $\overline{\lambda}(t) = 1$. An application of Theorem 6 gives

$$\liminf_{x\to+\infty} -\frac{2}{x^2}\log\big[\overline{p}(x) - p_M(x)\big] \ge 1 + 1/Z_\Delta. \tag{85}$$

where

$$Z_\Delta := \sup_{z\in(0,\Delta]} \frac{1-\rho^2(z^2) - 4\rho'^2(z^2)z^2}{[1-\rho(z^2)]^2} + \max_{z\in(0,\Delta]} \frac{4\big[\rho'(z^2)^+ z\big]^2}{[1-\rho(z^2)]^2},$$

and Δ is the diameter of S.

5) Suppose that

- the process $\mathcal{X}$ is stationary with covariance $\Gamma(t) := \mathrm{Cov}(X(s), X(s+t))$ that satisfies $\Gamma(s_1, \dots, s_d) = \prod_{i=1,\dots,d} \Gamma_i(s_i)$ where $\Gamma_1, ..., \Gamma_d$ are d covariance functions on $\mathbb{R}$ which are monotone, positive on $[0, +\infty)$ and of class $\mathcal{C}^4$,
- S is a rectangle
$$S = \prod_{i=1,\dots,d} [a_i, b_i] \quad , a_i < b_i.$$

Then, adding an appropriate non-degeneracy condition, conditions A2-A5 are fulfilled and Theorem 6 applies

It is easy to see that

$$-r_{0,1}(s,t) = \begin{bmatrix} \Gamma_1'(s_1-t_1)\Gamma_2(s_2-t_2)\dots\Gamma_d(s_d-t_d) \\ \vdots \\ \Gamma_1(s_1-t_1)\dots\Gamma_{d-1}(s_{d-1}-t_{d-1}).\Gamma_d'(s_d-t_d) \end{bmatrix}$$

belongs to $\mathcal{C}_{t,j}$ for every $s \in S$. As a consequence $\kappa_t = 0$ for all $t \in S$. On the other hand, standard regressions formulae show that

$$\frac{\mathrm{Var}\big(X(s)/X(t), X'(t)\big)}{(1-r(s,t))^2} = \frac{1 - \Gamma_1^2 \dots \Gamma_d^2 - \Gamma_1'^2\Gamma_2^2 \dots \Gamma_d^2 - \dots - \Gamma_1^2 \dots \Gamma_{d-1}^2 \Gamma_d'^2}{(1-\Gamma_1 \dots \Gamma_d)^2},$$

where Γ_i stands for $\Gamma_i(s_i - t_i)$. Computation and maximisation of σ_t^2 should be performed numerically in each particular case.

References

[1] Adler, R.J. (1981). *The Geometry of Random Fields.* Wiley, New York.

[2] Adler, R.J. (1990). *An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes.* IMS, Hayward, Ca.

[3] Adler, R.J. and Taylor J. E.(2005). *Random fields and geometry. Book to appear.*

[4] Azaïs J-M., Bardet J-M. and Wschebor M. (2002). On the Tails of the distribution of the maximum of a smooth stationary Gaussian Process. *Esaim: P. and S.*, 6,177-184.

[5] Azaïs, J-M. and Delmas, C. (2002). Asymptotic expansions for the distribution of the maximum of a Gaussian random fields. *Extremes* (2002)5(2), 181-212.

[6] Azaïs, J-M and Wschebor, M. (2002). The Distribution of the Maximum of a Gaussian Process: Rice Method Revisited, *In and out of equilibrium: probability with a physical flavour*, Progress in Probability, 321-348, Birkhaüser.

[7] Azaïs J-M. and Wschebor M (2001). On the regularity of the distribution of the Maximum of one parameter Gaussian processes *Probab. Theory Relat. Fields*, 119, 70-98.

[8] Azaïs J-M. and Wschebor M (2005). On the Distribution of the Maximum of a Gaussian Field with d Parameters. *Annals Applied Probability*, 15 (1A), 254-278.

[9] Azaïs J-M. and Wschebor, M. (2006). A self contained proof of the Rice formula for random fields. Preprint available at http://www.lsp.ups-tlse.fr/Azais/publi/completeproof.pdf.

[10] Belyaev, Y. (1966). On the number of intersections of a level by a Gaussian Stochastic process. *Theory Prob. Appl.*, 11, 106-113.

[11] Berman, S.M. (1985a). An asymptotic formula for the distribution of the maximum of a Gaussian process with stationary increments. *J. Appl. Prob.*, 22,454-460.

[12] Berman, S.M. (1992). *Sojourns and extremes of stochastic processes*, The Wadworth and Brooks, Probability Series.

[13] Borell, C. (1975). The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30, 207-216.

[14] Borell, C. (2003). The Ehrhard inequality. *C.R. Acad. Sci. Paris*, Sér. I, 337, 663-666.

[15] Cramér, H. and Leadbetter, M.R. (1967). *Stationary and Related Stochastic Processes*, J. Wiley & Sons, New-York.

[16] Cucker, F. and Wschebor M. (2003). On the Expected Condition Number of Linear Programming Problems, *Numer. Math.*, 94, 419-478.

[17] Fernique, X.(1975). Régularité des trajectoires des fonctions aléatoires gaussiennes. *Ecole d'Eté de Probabilités de Saint Flour (1974). Lecture Notes in Mathematics*, 480, Springer-Verlag, New-York.

[18] Fyodorov, Y. (2006). Complexity of Random Energy Landscapes, Glass Transition and Absolute Value of Spectral Determinant of Random Matrices *Physical Review Letters* v. 92 (2004), 240601 (4pages); Erratum: ibid. v.93 (2004),149901(1page)

[19] Kendall, M.G., Stuart,A. and Ord, J.K. (1987). *The Advanced Theory of Statistics*, Vol. 3.

[20] Landau, H.J. and Shepp, L.A (1970). On the supremum of a Gaussian process. *Sankya Ser. A* 32, 369-378.

[21] Ledoux, M. (2001). *The Concentration of Measure Phenomenon.* American Math. Soc., Providence, RI.

[22] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces*, Springer-Verlag, New-York.

[23] Marcus, M.B. (1977). Level Crossings of a Stochastic Process with Absolutely Continuous Sample Paths, *Ann. Probab.*, 5, 52-71.

[24] Marcus, M.B. and Shepp, L.A. (1972). Sample behaviour of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.*, 2, 423-442.

[25] Mehta,M.L. (2004). *Random matrices*, 3d-ed. Academic Press.

[26] Mercadier, C. (2006). Numerical bounds for the distribution of the maximum of one- and two-dimensional processes, to appear *in Advances in Applied Probability*, 38, (1).

[27] Piterbarg, V; I. (1981). Comparison of distribution functions of maxima of Gaussian processes. *Th, Proba. Appl.*, 26, 687-705.

[28] Piterbarg, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields.* American Mathematical Society. Providence. Rhode Island.

[29] Rychlik, I. (1990). New bounds for the first passage, wave-length and amplitude densities. *Stochastic Processes and their Applications*, 34, 313-339.

[30] Sudakov, V.N. and Tsirelson, B.S. (1974). Extremal properties of half spaces for spherically invariant measures (in Russian). *Zap. Nauchn. Sem. LOMI*, 45, 75-82.

[31] Sun, J. (1993). Tail Probabilities of the Maxima of Gaussian Random Fields, *Ann. Probab.*, 21, 34-71.

[32] Talagrand, M. (1996). Majorising measures: the general chaining. *Ann. Probab.*, 24, 1049-1103.

[33] Taylor, J.E. and Adler, R. J. (2003). Euler characteristics for Gaussian fields on manifolds. *Ann. Probab.*, 31, 533-563.

[34] Taylor J.E., Takemura A. and Adler R.J. (2005). Validity of the expected Euler Characteristic heuristic. *Ann. Probab.*, 33, 4, 1362-1396.

[35] Wschebor, M. (1985). *Surfaces aléatoires. Mesure géometrique des ensembles de niveau.* Lecture Notes in Mathematics, 1147, Springer-Verlag.